

Sphere Rényi entropies

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I give some scalar field theory calculations on a d -dimensional lune of arbitrary angle, evaluating, numerically, the effective action which is expressed as a simple quadrature, for conformal coupling. Using this, the entanglement and Rényi entropies are computed. Massive fields are also considered and a renormalisation to make the (one-loop) effective action vanish for infinite mass is suggested and used, not entirely successfully. However a universal coefficient is derived from the large mass expansion.

From the deformation of the corresponding lune result, I conjecture that the effective action on all manifolds with a simple conical singularity has an extremum when the singularity disappears.

For the round sphere, I show how to convert the quadrature form of the conformal Laplacian determinant into the more usual sum of Riemann ζ -functions (and $\log 2$).

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1. Introduction.

Rényi entropy is a (continuous) extrapolation of Shannon entropy and has proved to be useful in various situations. In particular, it has come up in the context of entanglement and a number of explicit calculations have appeared. Casini and Huerta, [1], gave some expressions for the coefficient of the (universal) logarithmically divergent term in the entanglement entropy for conformally invariant scalar fields on even spheres and, more recently, Klebanov *et al*, [2], have discussed odd spheres. In earlier work, I considered the same situations using, as my workhorse, the orbifolded sphere S^d/\mathbb{Z}_q which introduces a conical singularity of angle $2\pi/q$ onto the separating submanifold so as to allow the standard construction of Callan and Wilczek, [3], of the entanglement entropy,

$$S_E = -(1 + q\partial_q) W(q) \Big|_{q \rightarrow 1}, \quad (1)$$

to be applied. $W(q)$ is the effective action on the orbifolded sphere.

In the light of the recent activity, I wish to consider Rényi entropies, from the same viewpoint but with a different calculational evaluation. Before that, I give a further treatment of the computation of operator determinants on spheres and extend the analysis to massive fields. Apart from a few underlying manipulations, my results are numerical and presented mainly as graphs in sections 5, 6 and 7.

2. The geometry

I have referred to the singular manifold as an orbifolded sphere. More exactly, it is a fundamental domain for the rotational cyclic action of \mathbb{Z}_q on S^d . This manifold could be termed a *periodic lune* of angle $2\pi/q$. I now extend this to a lune of *any* angle, even bigger than 2π . If q is the inverse of an integer this gives a multiple covering of the sphere.

For quantum field theory purposes, I need the spectral properties of the Laplacian. The conformally invariant eigenvalues on the q -lune can be taken as the union of two sets, λ_N and λ_D where, as used before, *e.g.* [4],

$$\begin{aligned} \lambda_N &= (a + qn_1 + n_2 + \dots + n_d)^2 - \frac{1}{4}, \quad (n_i = 0, 1, \dots, \infty) \\ \lambda_D &= (a + qn_1 + n_2 + \dots + n_d)^2 - \frac{1}{4}, \quad (n_1 = 1, \dots, \infty; n_2, \dots, n_d = 0, 1, \dots, \infty), \end{aligned} \quad (2)$$

with $a = (d-1)/2$. These are the eigenvalues on the Neumann and Dirichlet lunes of angle π/q and follow by standard separation of variables (*e.g.* Gromes [5], Pockels, [6]) or by algebraic means if q is integral, eg [7]. Any degeneracy is a consequence of coincidences.

3. Even dimensions

My main interest is in odd dimensions but, for completeness, I repeat a few old things. In even dimensions, the effective action generally diverges. Conventionally, the coefficient of the associated logarithmic term, say g , provides a universal (*i.e.* regularisation independent) contribution to the entanglement entropy. This coefficient, $C_{d/2}(q)$, is, essentially, the conformal anomaly and the recipe (1) yields this part of the entanglement entropy as the derivative of the contribution to $C_{d/2}$ of the co-dimension 2's worth of conical singularity *i.e.* ,

$$g_{d/2} = -\partial_q \left(C_{d/2}(q) - \frac{1}{q} C_{d/2}(1) \right) \Big|_{q=1} . \quad (3)$$

The heat-kernel coefficient, $C_{d/2}(q)$, is the value of the relevant ζ -function, $\zeta(s, q)$, at $s = 0$. Manipulation reveals that each set of eigenvalues in (2) gives, for $\zeta(0, q)$, the (same) sum of two generalised Bernoulli polynomials the properties of which then show that the first term in (3) vanishes, *i.e.* the conformal anomaly on the lune has an extremum at the full sphere. This is the essential mathematical point of [8]. The conclusion is that the entanglement entropy is just the conformal anomaly.

Turning to the Rényi entropy, the definition is,

$$S_n = \frac{nW(1) - W(1/n)}{1 - n} , \quad (4)$$

where $W(q)$ is the effective action on the periodic q -lune. It can be seen that the logarithm coefficient in S_n is essentially just the bracket in (3), evaluated at the conventional covering values $q = 1/n$ (although q could be anything).

It is easy to compute the generalised Bernoulli polynomials by, say, iteration and these trivially yield the formulae in Casini and Huerta, [1], obtained in a different way. For the record, I give the expressions for the Rényi logarithm coefficients, $g_8(n)$ and $g_{10}(n)$,

$$g_8(n) = \frac{(n+1)(79n^6 + 79n^4 + 23n^2 + 3)}{1814400n^7}$$

$$g_{10}(n) = \frac{(n+1)(1759n^8 + 1759n^6 + 571n^4 + 109n^2 + 10)}{239500800n^9} .$$

As a check, setting n to 1 gives, of course, the standard numbers of the classic round conformal anomaly.²

4. Odd dimensions. Effective action.

For closed, odd-dimensional manifolds, in the absence of zero modes, there is no divergence in the effective action using ζ -function regularisation and there is no logarithm term. In particular, for conformally invariant propagation, the conformal anomaly vanishes. In this case, it was suggested early on by Ryu and Takayanagi, [11], that the constant term in the entropy (independent of any introduced cut-off) could be taken as the universal term. Myers and Sinha, [12], considered this viewpoint further in their search for a relevant c -theorem. As pointed out in [13], in ζ -function regularisation there are no divergences and the entire expression for the entropy should be considered universal. The formula (1) still stands, and now it is the effective action, $W(q)$, that is conformally invariant.

Before proceeding to novelties, I mention that, in [13], It was shown that the effective action, $W(q)$, had an extremum at the singularity-free, round point, $q = 1$, so that the entanglement entropy is just minus the round effective action. Much earlier calculations of this quantity could then be drawn upon for analytic forms and numerical values.

Quite recently, Klebanov *et al*, [2], have evaluated the Rényi entropies for the same situation. I will obtain the same results in a more numerical fashion, but one that allows extensions to be made. I restrict the discussion to scalars.

Technically I am assuming, as a working hypothesis, that the effective action is given as the mathematically definite quantity,

$$W(q) = -\frac{1}{2}\zeta'(0, q), \quad (5)$$

where the basic calculational device is the ζ -function constructed from the eigenvalues (2) which I can write as $\zeta(s, q) = \zeta_N(s, q) + \zeta_D(s, q)$. To encompass different situations, I define a slightly more general object,

$$\zeta(s, a, \alpha \mid \boldsymbol{\omega}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \frac{1}{((a + \mathbf{m} \cdot \boldsymbol{\omega})^2 - \alpha^2)^s}, \quad (6)$$

² The Bernoulli form of the round conformal anomaly could be avoided by using the integral expression derived holographically by Diaz, [9], and by direct manipulation in [10].

where \mathbf{m} and $\boldsymbol{\omega}$ are d -vectors, so that,

$$\begin{aligned}\zeta(s, q) &= \zeta\left(s, a, \frac{1}{2} \mid q, \mathbf{1}\right) + \zeta\left(s, a + q, \frac{1}{2} \mid q, \mathbf{1}\right), \quad a = (d-1)/2 \\ &= \zeta\left(s, a_N, \frac{1}{2} \mid q, \mathbf{1}\right) + \zeta\left(s, a_D, \frac{1}{2} \mid q, \mathbf{1}\right),\end{aligned}\tag{7}$$

where $\mathbf{1}$ is a $(d-1)$ -vector. I refer to the real numbers $\boldsymbol{\omega}$ as the *parameters*, and endeavour to keep them general as long as possible. Also, in the combination (7), I require only that $a_N + a_D = \sum_i \omega_i \equiv 2\omega$.

Incidentally, if the field is conformal in $(d+1)$ dimensions, then $\alpha = 0$, and $\zeta(s, a, \alpha; \boldsymbol{\omega})$ reduces to the more elegant Barnes function. In previous evaluations, *e.g.* [4], of the required quantity, $\zeta'(0, a, \alpha; \boldsymbol{\omega})$, an expansion in α sufficed and allowed the Barnes function to be employed. Here I wish to work directly with the form (6).

As mentioned in [14], following Candelas and Weinberg, [15], and Minakshisudaram, [16], I can employ the Bessel function form,

$$\zeta(s, a, \alpha \mid \boldsymbol{\omega}) = \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \frac{d\tau \exp(-a\tau)}{\prod_{i=1}^d (1 - \exp(-\omega_i \tau))} \left(\frac{\tau}{2\alpha}\right)^{s-1/2} I_{s-1/2}(\alpha\tau), \tag{8}$$

which exposes the cylinder kernel.

Adding the N and D parts of the ζ -function yields the combined expression,

$$\begin{aligned}\zeta(s, a, \alpha \mid \boldsymbol{\omega}) &= \frac{2\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \frac{d\tau \cosh((\omega - a)\tau)}{\prod_{i=1}^d 2 \sinh(\omega_i \tau/2)} \left(\frac{\tau}{2\alpha}\right)^{s-1/2} I_{s-1/2}(\alpha\tau) \\ &\equiv \frac{I(s)}{\Gamma(s)},\end{aligned}\tag{9}$$

and the object is to evaluate $\zeta'(0, a, \alpha \mid \boldsymbol{\omega})$ which necessitates a continuation of (9) to around $s = 0$. I adopt the procedure of Candelas and Weinberg, [15], as used by Chodos and Myers, [17]. First note that, because the conformal anomaly, $\zeta(0, a, \alpha \mid \boldsymbol{\omega})$, is zero, the integral, $I(s)$, is well behaved.³ Therefore the derivative at zero is just $I(0)$. Next remark that the integrand is of the form $\tau^{2s-1-d} f(\tau^2)$ so that $I(s)$ can be continued via the complex integral,

$$I(s) = \frac{2\sqrt{\pi}}{1 + e^{\pi i(2s-1-d)}} \int_{-\infty+i\Delta}^{\infty+i\Delta} \frac{d\tau \cosh((\omega - a)\tau)}{\prod_{i=1}^d 2 \sinh(\omega_i \tau/2)} \left(\frac{\tau}{2\alpha}\right)^{s-1/2} I_{s-1/2}(\alpha\tau), \tag{10}$$

³ I comment that the value of the ζ -function at negative integers also vanishes and that the derivative at these points is also available numerically as is $\zeta(s)$ at positive integers. Other values would require a complex treatment of Bessel functions.

in which s can be set to zero with impunity, so long as d is odd and Δ lies between zero and the first zero of the denominator (which lies on the imaginary axis) *i.e.* $\Delta < 2\pi/\max \omega_i$. (Δ could be chosen bigger, as long as any poles are allowed for.)

Doing this yields,

$$\begin{aligned} I(0) &= 2 \int_C \frac{d\tau \cosh(\omega - a)\tau \cosh \alpha\tau}{\tau \prod_{i=1}^d 2 \sinh(\omega_i \tau/2)} \\ &= \frac{1}{2^{d-2}} \int_0^\infty dx \operatorname{Re} \frac{\cosh(\omega - a)\tau \cosh \alpha\tau}{\tau \prod_{i=1}^d \sinh(\omega_i \tau/2)}, \quad \tau = x + i\Delta, \end{aligned} \quad (11)$$

which is a suitable case for numerical treatment for any ω , a and α , so long as it converges.

Conformal fields mean $\alpha = 1/2$. The d -dimensional q -lune is given by the choices, $a = (d-1)/2$ and $\omega = (q, \mathbf{1})$, so $\omega = (d-1+q)/2$ and $0 < \Delta < 2\pi/q$.

I note, for future use, that a massive scalar can be accommodated by setting $\alpha^2 = 1/4 - \mu^2$. A minimal scalar corresponds to $\alpha = (d-1)/2$ leading to an infra red divergence in the integral (11) for large τ caused by a zero mode which has now to be taken into account. I will not be concerned with this here.

If $q = 1$, *i.e.* the full round sphere, a convenient choice is $\Delta = \pi$, [15], for then the real part can be explicitly taken for all d and easy algebra produces,

$$\begin{aligned} I(0, d) &= \frac{(-1)^{(d-3)/2}}{2^{d-2}} \int_0^\infty dx \frac{\pi}{x^2 + \pi^2} (\operatorname{sech}^d x/2 - \operatorname{sech}^{d-2} x/2) \\ &= \frac{(-1)^{(d-3)/2}}{2^{d-2}} (J(d) - J(d-2)), \end{aligned} \quad (12)$$

where

$$J(d) \equiv \int_0^\infty dx \frac{1}{(x^2 + 1) \cosh^d \pi x/2}. \quad (13)$$

Equation (12) affords a simpler way of evaluating the effective action on the sphere than the sums of Riemann ζ -functions resulting from closing the contour in (11) in the upper half plane. These sums also follow directly from the manipulation of the eigenvalue form of the ζ -function, the method employed in the earlier calculations. A sample number is $I(0, 21) = 1.664755684 \cdot 10^{-8}$ obtained instantaneously from (12).⁴ As is clear from (12), the values decrease, oscillating with d about zero which is, perhaps, not so immediately obvious from the alternative forms.⁵ In the appendix, I provide a method of deriving the sum of ζ -functions form from (12).

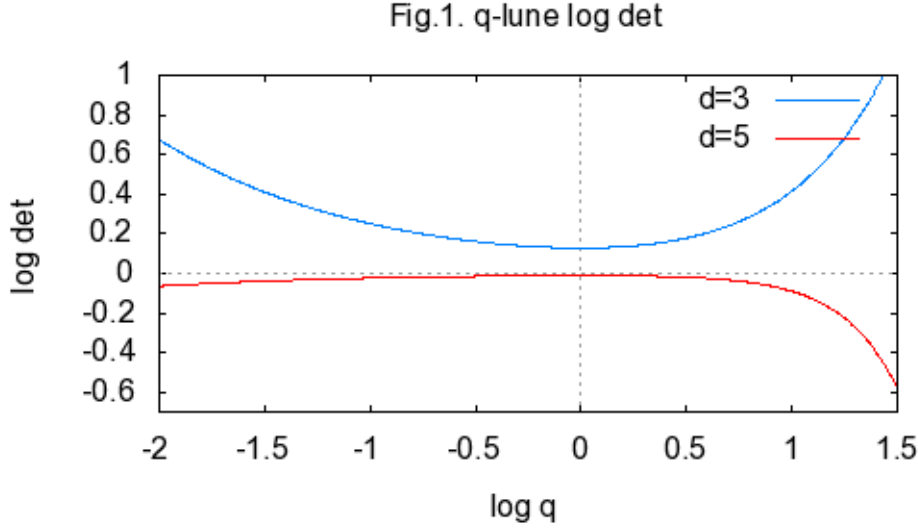
⁴The Riemann zeta form contains 11 terms. See the appendix.

⁵A different integral expression having the same sign factor was given in [13,18]. An integral form, obtained by conformal transformation to a hyperbolic cylinder, is given in [2] for three dimensions.

For any q , I pursue the simplest path and just compute the integral,

$$I(0, d, q) = \frac{1}{2^{d-2}} \int_0^\infty dx \operatorname{Re} \frac{\coth q\tau/2 \cosh \tau/2}{\tau \sinh^{d-1} \tau/2}, \quad (14)$$

for any $\Delta < 2\pi/q$ without further simplification. It is best to plot the results as functions of q and the results for the 3d and 5d lunes are given in Fig.1. As anticipated, and shown in [13], the effective action has an extremum at $q = 1$, the round sphere.



The proof of this fact in [13] utilised the explicit expression of the effective action in terms of Barnes ζ -functions and some specific properties of generalised Bernoulli polynomials. It can be seen here directly from the elementary circumstance that at $q = 1$ the derivative with respect to q of the integral in (14) is zero. There is actually no real part for $\Delta = \pi$.

I conjecture that this result is a particularly explicit case of a general feature of effective actions (functional determinants) on manifolds having a simple conical singularity of co-dimension 2 extent such as those, \mathcal{M}_β , investigated in [19] and [20], the statement being that the effective action has an extremum when the conical singularity fades away ($\beta \rightarrow 2\pi$). Indeed, it is difficult to imagine otherwise. If the lune geometry is deformed from spherical, still maintaining the geodesic embedding of the co-dimension 2 submanifold, it is hard to see the extremum disappearing, by continuity, and there is nothing to change its location.

5. Massive fields

Because massive fields are not conformally invariant, the significance of the lune geometry is not so apparent. However, because the calculation is straightforward, I will investigate an entanglement entropy in the massive case. For the moment, I maintain the assumption (5) for the effective action, which I refer to as the ‘bare’ expression.

Setting $\alpha^2 = 1/4 - \mu^2$, the mass parameter μ measures the deviation from conformal coupling and one now has the specialisation of (11), for massive fields on the lune,

$$I(0, d, q, \alpha) = \frac{1}{2^{d-2}} \int_0^\infty dx \operatorname{Re} \frac{\coth q\tau/2 \cosh \alpha\tau}{\tau \sinh^{d-1} \tau/2}, \quad (15)$$

which is, apparently, valid for the entire range of μ , $0 \leq \mu \leq \infty$ and can again be treated purely numerically. The asymptotic limit for large μ^2 can be used as a check. Details are given in the next section.

For the entropy the derivative term, (1), comes into play, the appropriate combination being,

$$\begin{aligned} \mathfrak{S}(d, \alpha) &\equiv \frac{1}{2} (I(0, d, 1, \alpha) + \partial_q I(0, d, q, \alpha)) \Big|_{q=1} \\ &= \frac{1}{2^d} \int_0^\infty dx \operatorname{Re} \left[\frac{2 \cosh \tau/2 \cosh \alpha\tau}{\tau \sinh^d \tau/2} - \frac{\cosh \alpha\tau}{\sinh^{d+1} \tau/2} \right]. \end{aligned} \quad (16)$$

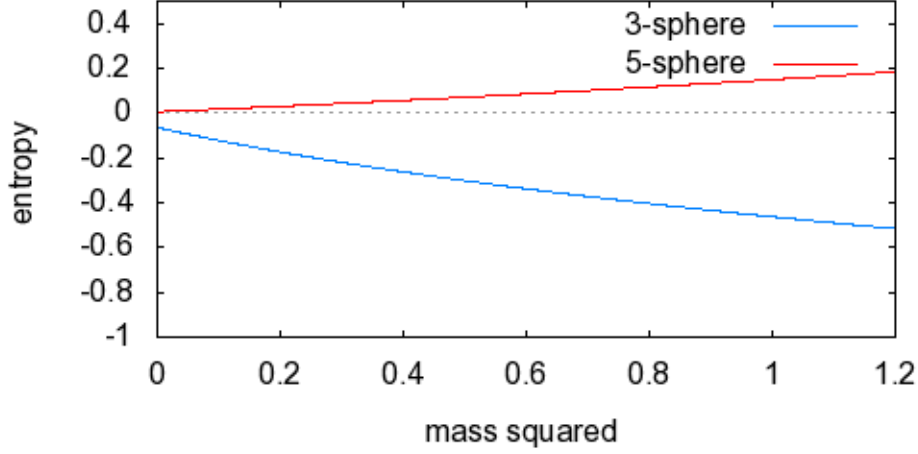
Plots of this entanglement entropy for the 3- and 5-spheres as functions of μ^2 are shown in Fig.2.

An important quantity is the derivative with respect to μ^2 at $\mu = 0$ which is, to a sign, the derivative with respect to α at $\alpha = 1/2$. I find,

$$\begin{aligned} \partial_\alpha \mathfrak{S}(d, \alpha) &= \Big|_{\alpha=1/2} = \frac{1}{2^d} \int_0^\infty dx \operatorname{Re} \left[\frac{2 \cosh \tau/2}{\sinh^{d-1} \tau/2} - \frac{\tau}{\sinh^d \tau/2} \right] \\ &= \frac{1}{2^d} \int_0^\infty dx \operatorname{Re} \left[\frac{\sinh \tau - \tau}{\sinh^d \tau/2} \right] \\ &= -(-1)^{(d-1)/2} \frac{\pi}{2^d} \int_0^\infty dx \frac{1}{\cosh^d x/2} \\ &= -(-1)^{(d-1)/2} \frac{\Gamma(d)}{2^{2d-2} \Gamma^2((d+1)/2)} \pi^2. \end{aligned}$$

For $d = 3$, this agrees with the value quoted in Klebanov *et al* ,[21], obtained by an unspecified method. The non-zero value shows that this entanglement entropy is non-stationary at the UV fixed point, *cf* [21].

Fig.2. Massive entanglement entropy



6. Asymptotic behaviour. Renormalisation.

Information is contained in the behaviour for large mass. The asymptotic expansion for the derivative of the ζ -function at zero is easily found from the definition and is, in its generic form, for odd d ,

$$\zeta'(0, m^2) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0,1/2,1,\dots}^{\infty} (m^2)^{d/2-n} \Gamma(n - d/2) C_n, \quad (17)$$

where the C_n are the short-time expansion heat-kernel coefficients for the operator $-\Delta + V$ if the actual propagation operator is $-\Delta + V + m^2$. In the present setting, it is most convenient algebraically to choose the addition, V , so that $-\Delta + V$ is conformally invariant in $d + 1$ dimensions. For spheres, and their factors, this has the consequence that the heat-kernel expansion terminates, [7]. For spheres, $V = ((d - 1)/2)^2$ and the propagation operator is, in our existing notation,

$$-\Delta + \left(\frac{d-1}{2}\right)^2 - \alpha^2,$$

so that $\alpha^2 = -m^2$.

The coefficients derived in [7] allow the asymptotic form, (17), to be made explicit,

$$I(0, d, q, im) \sim \frac{2\pi}{q} \sum_{k=1,3,\dots}^d \frac{(-1)^{(k+1)/2}}{k!(d-k)!} B_{d-k}^{(d)}((d-1)/2 \mid q, \mathbf{1}) m^k \quad (18)$$

in terms of generalised Bernoulli polynomials. Two examples should suffice

$$\begin{aligned} I(0, 3, q, im) &\sim \frac{\pi}{q} \left(\frac{1}{3} m^3 - \frac{q^2 - 1}{6} m \right) \\ I(0, 5, q, im) &\sim -\frac{\pi}{q} \left(\frac{1}{60} m^5 - \frac{q^2 - 2}{36} m^3 - \frac{(q^2 - 1)(q^2 + 11)}{360} m \right), \end{aligned} \quad (19)$$

and this is all. The final term always vanishes for the full round sphere, $q = 1$. The expressions furnish a useful test of the numerics and the limiting behaviour can be seen, roughly, in Fig.1. Because of the termination, the errors in the forms (19) are exponentially small, a desirable feature.

The divergence as m tends to infinity brings into focus the relation between the logdet and the effective action since the latter is expected to be zero in the infinite mass limit, on physical grounds. Hence I could take the right-hand side of (18) over to the left hand side and consider this to be a finitely renormalised (minus) logdet. This is equivalent to the renormalisation procedure advocated by De Witt, [22], and amounts to subtracting from the propagation heat-kernel, $K(\tau)$, sufficient terms of its short-time expansion so as to render the formal expression for the effective action,

$$W = \int_0^\infty d\tau \frac{K(\tau)}{\tau} + \text{const},$$

finite. (See [22], equn. (14.75).)

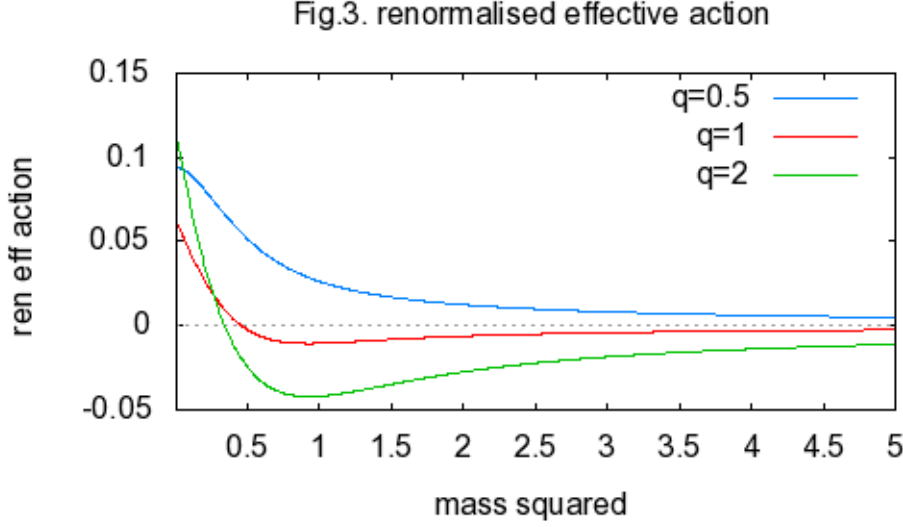
The problem with this is that the so renormalised effective action has to be evaluated at the conformal point, $\alpha^2 = 1/4$ *i.e.* at $m^2 = -1/4$ where it is non-zero and, worse, formally imaginary. Hence I will make a further, *ad hoc* hypothesis and require the subtraction term to vanish at this point, in addition to having the asymptotic form, (18). This can be accomplished by shifting the mass to $\mu^2 = 1/4 - \alpha^2$ which vanishes at the conformal value. The term to be subtracted from $\zeta'(0, q)$ is then in three dimensions,

$$\frac{1}{6\pi} \left(\mu^3 C_0 - \mu \frac{3}{2} (C_1 + \frac{1}{4} C_0) \right),$$

which for the three dimensional q -lune is to be compared with (19) with $C_0 = 2\pi^2/q$ and $C_1 = 2\pi^2(q^2 - 1)/3q$. The renormalised massive effective action for this case is thus *defined* as,

$$\begin{aligned} W_{ren}(q, \mu) &= -\frac{1}{2} \zeta'(0, q) + \frac{\pi}{6q} \left(\mu^3 - \frac{1}{8} \mu (4q^2 - 1) \right) \\ &= -\frac{1}{2} I(0, 3, q, \alpha) + \frac{\pi}{6q} \left(\mu^3 - \frac{1}{8} \mu (4q^2 - 1) \right), \end{aligned} \quad (20)$$

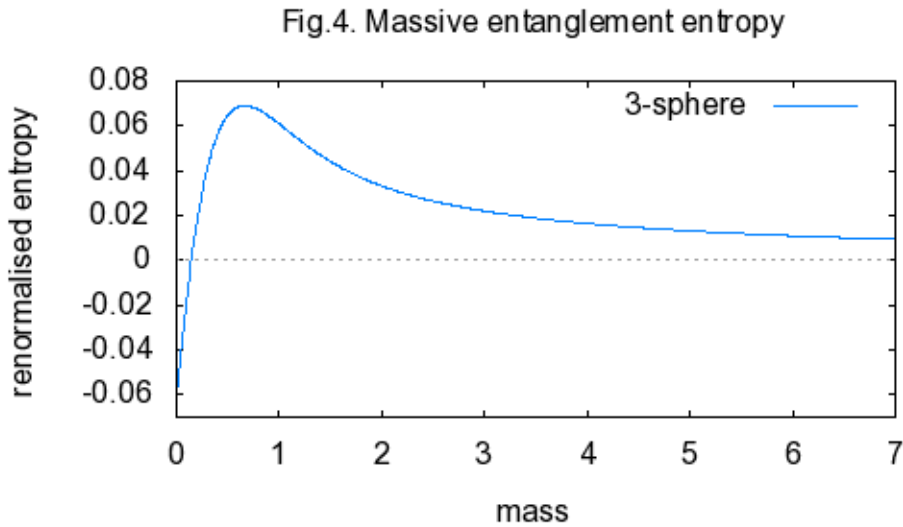
where $I(0)$ is given by (15) with $\alpha^2 = 1/4 - \mu^2$. A few plots of this effective action are presented in Fig.3.



This renormalisation procedure can at best be considered makeshift. An advantage is that the vanishing at the conformal point means that all the preceding conformal arguments hold true, including the extremum at $q = 1$. However, the infinite μ behaviour is not exponentially vanishing. Because of this I do not discuss the higher spheres, but I will give the resulting form of the entropy for three dimensions. The renormalisation modifies the bare entanglement entropy $\mathfrak{S}(d, \alpha)$,

$$\mathfrak{S}_{ren}(3, \alpha) = \mathfrak{S}(3, \alpha) + \frac{\pi\mu}{6}, \quad (\alpha^2 = 1/4 - \mu^2),$$

for $d = 3$ and a plot, against μ , is given in Fig.4.



7. Rényi entropies

In Fig.5, I plot the conformal Rényi entropy, S_n , (4), versus a continuous $n \equiv 1/q$ for the 3- and 5-spheres.

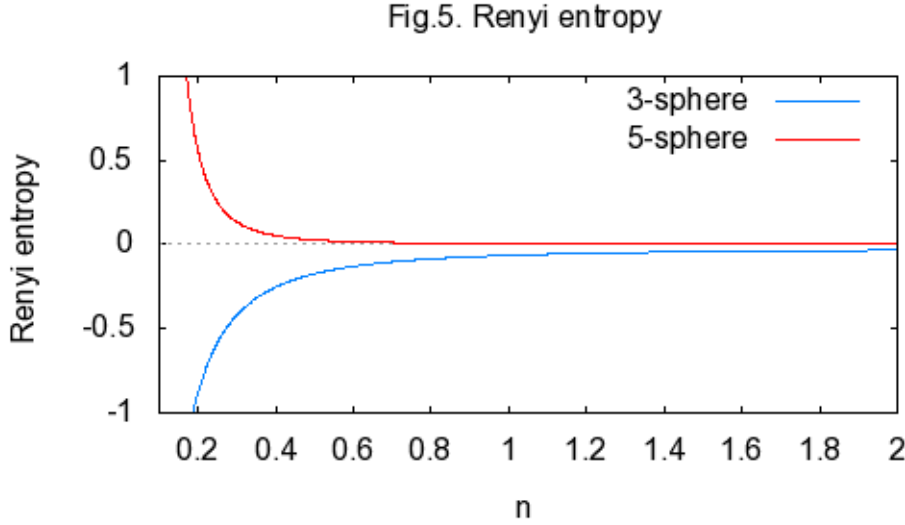
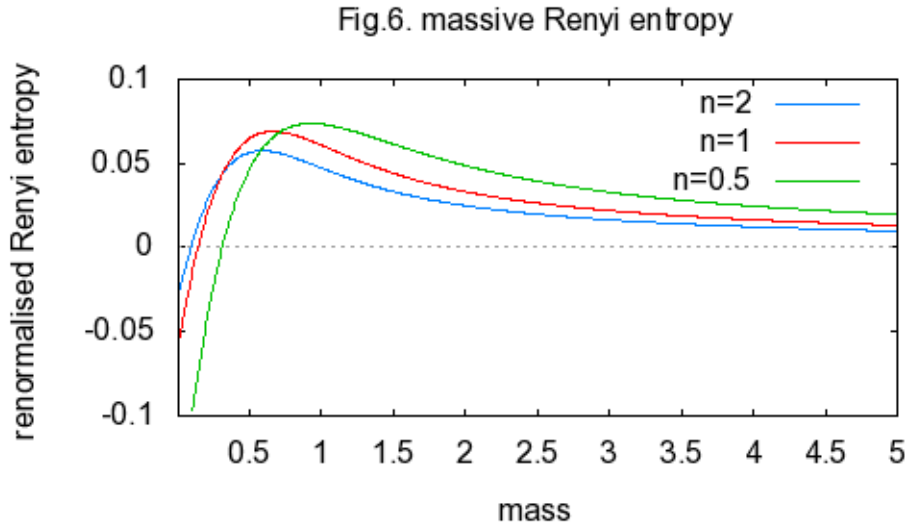


Fig. 6 is for the 3-sphere only, and shows the variation of the renormalised Rényi entropy with mass μ . Using (20), this is given, for $d = 3$, by

$$S_n^{ren} = S_n + \frac{\pi}{12} \left(\frac{1+n}{n} \right) \mu. \quad (21)$$



The coefficient of the linear asymptotic behaviour of S_n with mass (cancelled in (21)) is expected to be a universal constant (see [23]). For arbitrary dimensions,

the form, (18), allows one to find this constant as the coefficient of the highest power of m (or μ), which is m^{d-2} . Easy algebra yields for this coefficient,

$$r_n(d) = (-1)^{(d-1)/2} \frac{\pi}{12(d-2)!} \frac{n+1}{n},$$

for the present toy model. This should be compared with the similar Gaussian expression obtained in [23], *cf* also Calabrese and Cardy, [24].

Setting $n = 1$ gives the entanglement entropy constant,

$$r_1(d) = (-1)^{(d-1)/2} \frac{\pi}{6(d-2)!}, \quad (22)$$

which could be compared with the formula of Hertzberg and Wilczek, [25], derived in basically the same way, but for a different geometry. Their expression, on taking the dividing surface to be a $(d-2)$ -sphere, turns out to be identical to (22) showing that the curvature makes no difference to the result which is, not surprisingly, due just to the local conical singularity.

Hertzberg, [26], has analysed the dependence of the entanglement entropy on the mass for an interacting field theory. To leading order, the result is the replacement of the bare mass in the free field expression by its renormalised value.

The (finite number of) lower powers of the mass in the asymptotic form are curvature effects. This is in concordance with the observations of Lewkowycz, Myers and Smolkin, [27], who generalise the geometry of [25] to a waveguide of curved cross section.

8. Discussion

I have presented a more numerical approach to the computation of the conformal effective action, and related quantities, on an odd-dimensional lune of arbitrary angle. For the full, round sphere this leads to yet another expression for the Laplacian determinant. In particular I thence obtain the older form of a sum of Riemann ζ -functions at odd arguments.

The extension to a massive field is made and I attempt to renormalise the effective action so that it vanishes for infinite mass. The renormalisation implemented in section 6, which forces the effective action to vanish also at the conformal point, is suspect for small values of the mass, however I could not devise anything better.

Going beyond the lune, the numerical quadrature expression, (11), allows the log determinants on other, special spherical regions to be computed by an appropriate choice of the parameters, ω . For example, the conformal log determinant

on the periodic tesseract (Schläfli symbol [3,3,4] with $\omega = (4, 6, 8)$) is 0.579184 and will be discussed at another time. Related is the computation of Barnes' multiple Gamma function, depending on a and ω , which is quite hard to find.

Appendix. Expression in terms of Riemann ζ -function.

For those who like such things, I outline a way of transforming the integral expression (12) into the alternative form of the effective action on the full sphere as a sum of Riemann ζ -functions evaluated at positive odd integers. I also include some ancillary mathematical remarks.

The basic notion is to expand the powers of $\text{sech } x/2$ in multiple even derivatives of $\text{sech } x/2$. For odd $d = 2r + 1$ this is clearly possible ⁶ and I define coefficients, \mathcal{E}_ρ^r , by,

$$\text{sech}^{2r+1} x/2 = \sum_{\rho=0}^r \mathcal{E}_\rho^r \frac{d^{2\rho}}{dx^{2\rho}} \text{sech } x/2, \quad (23)$$

so that, from (13), one requires the integral,

$$\begin{aligned} K(\rho) &\equiv \frac{1}{\pi^{2\rho}} \int_0^\infty dx \frac{1}{(1+x^2)} \frac{d^{2\rho}}{dx^{2\rho}} \text{sech } \pi x/2 \\ &= \frac{1}{\pi^{2\rho}} \int_0^\infty dx \frac{d^{2\rho}}{dx^{2\rho}} \frac{1}{(1+x^2)} \text{sech } \pi x/2 \\ &= \frac{(2\rho)!}{\pi^{2\rho}} \int_0^\infty dx \frac{\sin(2\rho+1)\theta}{(1+x^2)^{2\rho+1}} \text{sech } \pi x/2, \end{aligned} \quad (24)$$

where I have used Liouville's form of the multiple derivative (see Gregory, [29], p.17, no.(20)) and $\theta = \tan^{-1} 1/x$. The oddness of the dimension (evenness of the derivative) means that $\sin(2\rho+1)\theta = (-1)^\rho \cos((2\rho+1)\tan^{-1} x)$ and at this point I recall an integral form for the Riemann ζ -function due to Jensen which, at odd arguments, yields, ⁷

$$\zeta_R(2\rho+1) = \frac{2^{2\rho}}{1-2^{-2\rho}} \int_0^\infty \frac{1}{(1+x^2)^{2\rho+1}} \frac{\cos((2\rho+1)\tan^{-1} x)}{\cosh \pi x/2}.$$

⁶Something related, actually the inverse, is performed by Stern, [28].

⁷ This formula is obtained in Lindelöf, [30], using a variant of Plana-Abel summation applied to the function $1/z^s$ and it may be a more uniform complex treatment of the present identities can be found.

It is therefore seen that the quantity $J(d)$, from which the derivative at zero, $I(0, d)$, can be found using (12), is given by,

$$\begin{aligned} J(2r+1) &= \sum_{\rho=0}^r \mathcal{E}_\rho^r K(\rho) \\ &= \sum_{\rho=1}^r (-1)^\rho \mathcal{E}_\rho^r \frac{(2\rho)!}{\pi^{2\rho}} \frac{1-2^{-2\rho}}{2^{2\rho}} \zeta_R(2\rho+1) + \mathcal{E}_0^r \log 2, \end{aligned}$$

which exhibits the usual structure.⁸ The end value $J(1)$, (associated with the pole of the Riemann ζ -function) is a special case and (*e.g.* Gregory, [29], p.496 example (d)),

$$J(1) = \int_0^\infty dx \frac{1}{(x^2+1) \cosh \pi x/2} = \log 2.$$

Incidentally, a more general integral is, [31], Table 97, No.4,

$$\begin{aligned} \int_0^\infty dx \frac{2a}{(x^2+a^2) \cosh \pi x/2} &= \psi\left(\frac{a+3}{4}\right) - \psi\left(\frac{a+1}{4}\right) = 2\beta\left(\frac{a+1}{2}\right) \\ &= 2 \int_0^1 dz \frac{z^{(a-1)/2}}{1+z}, \end{aligned}$$

in terms of Stirling's β function (*e.g.* Nielsen, [32], pp.16,181) and Legendre's integral form.

There is no simple expression for the rational numbers \mathcal{E}_ρ^r but they can be rapidly computed from the recursion, somewhat similar to that for generalised Euler numbers,

$$\mathcal{E}_\rho^r = \frac{(2r-1)}{2r} \mathcal{E}_\rho^{r-1} - \frac{2}{r(2r-1)} \mathcal{E}_{\rho-1}^{r-1}, \quad (25)$$

together with $\mathcal{E}_{-1}^r = 0 = \mathcal{E}_{r+1}^r$ and the initial value $\mathcal{E}_0^0 = 1$.

Expressions for the effective action in this particular form can be found listed in [33] up to $d = 11$. Out of interest I present the vector of the coefficients of $\zeta_R(2\rho+1)/2^{23}\pi^{2\rho}$ for ρ from 1 to 6 (*i.e.* $d = 13$),

$$\left[\frac{2385868}{17325}, \frac{21914}{135}, -\frac{6932}{15}, -\frac{7174}{3}, -4774, -4095 \right].$$

The coefficient of $\log 2$ is $21 \cdot 2^{-21}$.

⁸ Instead of the ζ -functions at odd positive numbers, the derivatives at negative even integers can be used. This results in a certain cosmetic improvement.

The numerical evaluation of these expressions can easily be automated, but is not so efficient as the earlier one using quadratures.

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